

# DECOMPOSITION OF TENSOR PRODUCTS OF MODULAR IRREDUCIBLE REPRESENTATIONS FOR $\mathrm{SL}_3$ : THE $p \geq 5$ CASE

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**ABSTRACT.** We study the structure of the indecomposable direct summands of tensor products of two *restricted* simple  $\mathrm{SL}_3(K)$ -modules, where  $K$  is an algebraically closed field of characteristic  $p \geq 5$ . We give a characteristic-free algorithm for the computation of the decomposition of such a tensor product into indecomposable modules.

The  $p < 5$  case for  $\mathrm{SL}_3(K)$  was studied in the authors' earlier paper [4]. In this paper we show that for characteristics  $p \geq 5$  all the indecomposable summands are rigid, in contrast to the situation in characteristic 3.

## 1. INTRODUCTION

Let  $G = \mathrm{SL}_3(K)$  where  $K$  is an algebraically closed field of characteristic  $p > 0$ . The purpose of this paper is to describe the indecomposable direct summands of a tensor product  $L \otimes L'$  of two simple  $G$ -modules  $L, L'$  of  $p$ -restricted highest weights. We give a characteristic-free algorithm for the computation of the decomposition of such a tensor product into indecomposable modules. Thanks to Steinberg's tensor product theorem, such data gives a first approximation toward a description of the indecomposable direct summands of a general tensor product of two (not necessarily restricted) simple  $G$ -modules.

Such questions were previously studied in [11] for  $\mathrm{SL}_2(K)$  and [4] for  $\mathrm{SL}_3(K)$  in the case  $p < 5$ . The present paper, which is a continuation of [4], concentrates on  $G = \mathrm{SL}_3(K)$  for  $p \geq 5$ . The papers [11, 4] contain sharper results than in the present paper; in particular the multiplicities of the indecomposable summands were explicitly determined for restricted tensor products, and results on the unrestricted problem were obtained as well. Our main focusses in the present paper are: to generalise the  $p$ -restricted algorithm in [4] to a characteristic-free description for all primes via the Littlewood–Richardson rule; to describe the structure of the indecomposable summands for  $p \geq 5$  in a characteristic-free way (in the restricted case) in terms of facet geometry.

Our main results are summarized in Theorems 1 and 2 in Section 6 and the aforementioned algorithm outlined in Section 8.4. In particular, unlike the situation in characteristic  $p = 3$ , in characteristics  $p \geq 5$  we find that all the summands are rigid modules (socle series and radical series coincide). All of the summands are in fact tilting modules, except for certain non-tilting simple modules and a certain family of non highest weight modules, which had also been observed in the  $p = 3$  case. The first examples of non-rigid tilting modules for algebraic groups were exhibited in [4]; further examples and a general positive rigidity result for tilting modules are now available in [2].

We also give a detailed analysis of the smallest tilting module of  $\mathrm{SL}_3$  exhibiting a structural symmetry arising from the type  $A_2$  Dynkin diagram automorphism. In [18] it is shown

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that any tilting module for a path algebra has a coefficient quiver such that all the coefficients can be chosen to be 1. There is no theory in place to tackle the problem for tilting modules of quotients of path algebras. We provide all possible bases of coefficient quivers for this tilting module, and show that there exist two distinct bases where all coefficients can be chosen to be equal to 1. This is, to our knowledge, the first non-trivial example of such a coefficient quiver for a tilting module of an algebraic group.

We had hoped that determining the indecomposable summands of  $L \otimes L'$  in the restricted case would lead to their determination in general, by some sort of generalised tensor product result; e.g., see Lemma 1.1 in [4]. However, our results show this is not the case, and the general (unrestricted) decomposition problem remains open. Although our results do in principle give a partial decomposition in the unrestricted case, using formula (1.1.3) of [4], the summands there will not always be indecomposable, and the problem of finding all splittings of those summands remains in general unsolved.

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## 2. NOTATION AND GENERAL FACTS

2.1. We fix our notation, following [15]. For a given semisimple simply-connected affine algebraic group  $G$  over an algebraically closed field  $K$ , we fix a maximal torus  $T \subset G$  and a Borel subgroup  $B$  containing  $T$ , and let  $B^+$  be the Borel subgroup opposite to  $B$ . We have the following standard notations:

- $h$  = the Coxeter number of  $G$ ;
- $X = X(T)$  = character group of  $T$ ;
- $Y = Y(T)$  = cocharacter group of  $T$ ;
- $R = R(G, T)$  = root system of the pair  $G, T$ ;
- $\gamma^\vee \in Y$  = coroot associated to a root  $\gamma \in R$ ;
- $R^+ = R(B^+, T)$  = set of positive roots;
- $S$  = the simple roots in  $R^+$ ;
- $W = N_G(T)/T$  = the Weyl group;
- $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$
- $X^+ = \{\lambda \in X : 0 \leq \langle \lambda, \alpha^\vee \rangle, \text{ all } \alpha \in S\}$  = the set of dominant weights;
- $X_r = \{\lambda \in X : 0 \leq \langle \lambda, \alpha^\vee \rangle < p^r, \text{ all } \alpha \in S\}$  (if characteristic  $K = p > 0$ )

which will be used throughout the paper. We define a partial order on the character group  $X = X(T)$  by letting  $\lambda \leq \mu \iff \mu - \lambda \in \mathbb{N}S = \sum_{\alpha \in S} \mathbb{N}\alpha$ . (Here  $\mathbb{N}$  denotes the set of nonnegative integers.)

2.2. Henceforth we shall always assume the underlying field  $K$  has characteristic  $p > 0$ . By a left “ $G$ -module” we always mean a *rational* left  $G$ -module, which is the same as a right  $K[G]$ -comodule. For each dominant weight  $\lambda \in X^+$  we have the following finite dimensional  $G$ -modules, which are uniquely determined up to isomorphism by their highest weight:

- $L(\lambda)$  = the simple module of highest weight  $\lambda$ ;
- $\Delta(\lambda)$  = the Weyl module of highest weight  $\lambda$ ;
- $\nabla(\lambda) = \text{ind}_B^G(K_\lambda)$  = the dual Weyl module of highest weight  $\lambda$ ;
- $T(\lambda)$  = the indecomposable tilting module of highest weight  $\lambda$ ;

$\text{St} = L((p-1)\rho) = \nabla((p-1)\rho) =$  the Steinberg module

where  $K_\lambda$  is the 1-dimensional  $B$ -module upon which  $T$  acts by the character  $\lambda$ , with the unipotent radical of  $B$  acting trivially. Following [4] we shall let  $\chi(\lambda)$  denote the (formal) character of the Weyl module of highest weight  $\lambda \in X^+$  and let  $\chi_p(\lambda)$  denote the character of the simple module of highest weight  $\lambda \in X$ .

The simple modules  $L(\lambda)$  are contravariantly self-dual. The module  $\nabla(\lambda)$  has simple socle isomorphic to  $L(\lambda)$  and the module  $\Delta(\lambda)$  is isomorphic to  ${}^\tau\nabla(\lambda)$ , the contravariant dual of  $\nabla(\lambda)$ , hence has simple head isomorphic to  $L(\lambda)$ . In case  $\nabla(\lambda) = L(\lambda)$  is simple, we have  $T(\lambda) = \Delta(\lambda) = \nabla(\lambda) = L(\lambda)$ ; this applies in particular to the Steinberg module  $\text{St}$ . We note that the indecomposable tilting modules  $T(\lambda)$  are always contravariantly self-dual, and a tensor product of two tilting modules is again tilting.

We say that a module has a *good filtration* if it can be filtered by  $\nabla$ -modules and a *Weyl filtration* if it can be filtered by  $\Delta$ -modules. A tilting module is a module with both a good filtration and a Weyl filtration. We let  $\mathcal{F}(\Delta)$  denote the full subcategory of the category of  $G$ -modules with objects the finite dimensional  $G$ -modules admitting a  $\Delta$ -filtration, and dually let  $\mathcal{F}(\nabla)$  denote the full subcategory with objects the finite dimensional  $G$ -modules admitting a  $\nabla$ -filtration.

2.3. Let  $G_1$  denote the kernel of the Frobenius  $p$ -endomorphism of  $G$ , and let  $G_1T$  denote the inverse image of  $T$  under the same map. By  $\hat{Q}_1(\lambda)$  we denote the  $G_1T$ -injective hull of  $L(\lambda)$  for any  $\lambda \in X_1$ . If  $p \geq 2h - 2$  then  $\hat{Q}_1(\mu)$  has for any  $\mu \in X_1$  a  $G$ -module structure; this structure is unique in the sense that any two such  $G$ -module structures are equivalent. (These statements are expected to hold for all  $p$ ; the validity of this expectation is well known in case  $G = \text{SL}_2(K)$  or  $\text{SL}_3(K)$ .)

2.4. Let  $M$  be a  $G$ -module. The socle of  $M$ , denoted by  $\text{soc } M$ , is defined to be the largest semisimple submodule of  $M$ . Now consider the socle of  $M/\text{soc } M$ , and let  $\text{soc}^2 M$  denote the submodule of  $M$  containing  $\text{soc } M$  such that  $\text{soc}^2 M/\text{soc } M$  is isomorphic to the socle of  $M/\text{soc } M$ . We then inductively define the socle series  $\{\text{soc}^i M\}$  so that  $\text{soc}^{i+1} M/\text{soc}^i M$  is isomorphic to the socle of  $\text{soc}^i M/\text{soc}^{i-1} M$ . Clearly we have

$$0 \subset \text{soc } M \subset \text{soc}^2 M \subset \cdots \subset \text{soc}^i M \subset \text{soc}^{i+1} M \subset \cdots \subset M$$

and if  $M$  has finitely many composition factors then there is some  $r$  such that  $\text{soc}^r M = M$ . The socle layers are the semisimple quotients  $\text{soc}_i M = \text{soc}^i M/\text{soc}^{i-1} M$ , where we define  $\text{soc}^0 M = 0$ .

The radical of  $M$ , denoted by  $\text{rad } M$ , is defined similarly to be the smallest submodule of  $M$  such that the corresponding quotient is semisimple. We then let  $\text{rad}^2 M = \text{rad}(\text{rad } M)$  and inductively define the radical series,  $\{\text{rad}^i M\}$ , of  $M$  by  $\text{rad}^{i+1} M = \text{rad}(\text{rad}^i M)$ . We have

$$M \supset \text{rad } M \supset \text{rad}^2 M \supset \cdots \supset \text{rad}^i M \supset \text{rad}^{i+1} M \supset \cdots \supset 0$$

and if  $M$  has finitely many composition factors then there is some  $s$  such that  $\text{rad}^s M = 0$ . The radical layers are the semisimple quotients  $\text{rad}_i M = \text{rad}^{i-1} M/\text{rad}^i M$ , where  $\text{rad}^0 M = M$ .

In particular, we let  $\text{Head } M$  denote the first radical layer  $\text{rad}_1 M$ . If  $M$  has a finite composition series, the length  $s$  of the radical series coincides with the length  $r$  of the socle series; this common number is called the Loewy length of  $M$  and denoted by  $ll(M)$ . A

module  $M$  is *rigid* if its radical and socle series coincide, i.e. if  $\text{rad}_i M \cong \text{soc}_{u(M)-i} M$  for all  $i$ .

In general, a module for a finite dimensional algebra is *local* if it is isomorphic to a homomorphic image of a projective cover of a simple module, and *co-local* if it is isomorphic to a submodule of the injective hull of a simple module.

2.5. We will often regard  $G$ -modules as modules for some (generalised) Schur algebra  $S(\pi)$  where  $\pi$  is a finite saturated poset of dominant weights; see [8] or Chapter II.A in [15] for details on generalised Schur algebras. For a given  $\lambda \in X^+$  we frequently use the notation  $S(\leq \lambda)$  to mean the generalised Schur algebra corresponding to the poset  $\pi = \{\mu \in X^+ : \mu \leq \lambda\}$ . Focussing on generalised Schur algebras allows us to use methods from the representations of finite dimensional algebras in calculations.

For any  $S(\pi)$  we let  $P(\lambda)$  denote the projective cover of  $L(\lambda)$  in the category of  $S(\pi)$ -modules. We note that the contravariant dual of  $P(\lambda)$  is isomorphic to the injective hull of  $L(\lambda)$  in the category of  $S(\pi)$ -modules.

The fact that  $S(\pi)$  is quasihereditary is of great value for the calculations of this paper, and used repeatedly, especially the following basic fact which is often called ‘‘Brauer–Humphreys reciprocity’’.

$$(2.5.1) \quad [P(\mu) : \Delta(\lambda)] = [\nabla(\lambda) : L(\mu)]$$

for all  $\lambda, \mu \in \pi$ . The left hand side in the equality is interpreted as the number of subquotients isomorphic to  $\Delta(\mu)$  in a  $\Delta$ -filtration of  $P(\lambda)$ ; this number is known to be independent of the choice of  $\Delta$ -filtration. The right hand side denotes the composition factor multiplicity, as usual. For a proof of the displayed equality, see Proposition A2.2(iv) in [9]. (The original proof of this fact goes back to Theorem 2.6 in [7].)

The Schur algebra setting also allows us to make use of the following refinement of (2.5.1) from [5], where it is proved to hold more generally for any quasi-hereditary algebra with a contravariant duality fixing the simple modules.

**Theorem.** *Let  $S(\pi)$  denote a Schur algebra corresponding to a finite saturated set  $\pi$  of dominant weights. For weights  $\lambda, \mu \in \pi$  we have the following reciprocity:*

$$[\text{rad}_i P(\mu) : L(\lambda)] = [\text{rad}_i P(\lambda) : L(\mu)].$$

*This reciprocity respects the  $\Delta$ -filtration of the projective modules:*

$$[\text{rad}_i P(\mu) : \text{Head } \Delta(\lambda)] = [\text{rad}_i \Delta(\lambda) : L(\mu)].$$

Here, by  $[\text{rad}_i P(\mu) : \text{Head } \Delta(\lambda)]$  we mean the number of successive quotients  $\Delta(\lambda_j)$  in a *fixed*  $\Delta$ -filtration of  $P(\mu)$  such that  $\lambda_j = \lambda$  and there is a surjection  $\text{rad}^i P(\mu) \rightarrow \Delta(\lambda)$  which carries the subquotient  $\Delta(\lambda_j)$  onto  $\Delta(\lambda)$ . One may easily check that this is independent of the choice of  $\Delta$ -filtration. We provide an example in subsection 7.1.

*Remark.* The contravariant dual of the above theorem relates the socle layers of injective modules, and gives information about where  $\nabla$ -modules occur in a  $\nabla$ -filtration of an injective module.

2.6. We will need a couple of other standard results from the theory of quasi-hereditary algebras, which are special cases of Proposition A2.2 of [9].

**Proposition.** *Let  $\lambda, \mu \in X^+$ , we have that:*

- (a) *If  $\text{Ext}^1(\Delta(\lambda), \Delta(\mu)) \neq 0$  then  $\mu \geq \lambda$ .*
- (b)  *$\dim_K \text{Hom}_G(X, Y) = \sum_{\nu \in X^+} [X : \Delta(\nu)][Y : \nabla(\nu)]$ , for any  $X \in \mathcal{F}(\Delta)$ ,  $Y \in \mathcal{F}(\nabla)$ .*

2.7. Assume that the field  $K$  is of characteristic  $p \geq 2h - 2$ . Thanks to results of Jantzen in [16] we have an isomorphism of  $S(\leq 2(p-1)\rho + w_0\lambda)$ -modules

$$T(2(p-1)\rho + w_0\lambda) \cong P(\lambda)$$

for any  $\lambda \in X_1$ , where  $w_0$  is the longest element in the Weyl group. This fact will be used repeatedly in the proof of our results. Note that any projective tilting module is also injective, since tilting modules are contravariantly self-dual, so the above module is projective, injective, and tilting, for any  $\lambda \in X_1$ .

2.8. There is a twisted tensor product theorem for tilting modules, assuming that Donkin's conjecture [10, (2.2) Conjecture] is valid. (It is well known [15, II.11.16, Remark 2] that the conjecture is valid for all  $p$  in case  $G = \text{SL}_3$ .) For our purposes, it is convenient to reformulate the tensor product theorem in the following form. First we observe that, given  $\lambda \in X^+$  satisfying the condition

$$(2.8.1) \quad \langle \lambda, \alpha^\vee \rangle \geq p-1, \quad \text{for all simple roots } \alpha,$$

we have

$$(2.8.2) \quad \lambda = \lambda' + p\mu, \quad \lambda' \in (p-1)\rho + X_1, \quad \mu \in X^+$$

where  $\lambda'$  and  $\mu$  are uniquely determined by the given conditions. By induction on  $m$  using (2.8.1) and (2.8.2) one shows that every  $\lambda \in X^+$  has a unique expression in the form

$$(2.8.3) \quad \lambda = \sum_{j=0}^m a_j(\lambda) p^j$$

with  $a_0(\lambda), \dots, a_{m-1}(\lambda) \in (p-1)\rho + X_1$  and  $\langle a_m(\lambda), \alpha^\vee \rangle < p-1$  for at least one simple root  $\alpha$ .

Given  $\lambda \in X^+$ , express  $\lambda$  in the form (2.8.3). Assume Donkin's conjecture holds: i.e., assume that  $T(\mu)$  is indecomposable on restriction to  $G_1$  for any  $\mu \in (p-1)\rho + X_1$ . Then there is an isomorphism of  $G$ -modules

$$(2.8.4) \quad T(\lambda) \simeq \bigotimes_{j=0}^m T(a_j(\lambda))^{[j]}.$$

To prove this one uses induction and [15, Lemma II.E.9] (which is a slight reformulation of [10, (2.1) Proposition]).

### 3. COEFFICIENT QUIVERS

The following is taken from [18]. Let  $K$  be a field, let  $Q$  be a (finite) quiver and  $KQ$  the path algebra of  $Q$  over  $K$ . Recall that a representation  $N$  of  $Q$  over  $K$  is of the form  $N = (N_x; N_\alpha)_{x, \alpha}$ ; here, for every vertex  $x$  of  $Q$ , there is given a finite-dimensional  $K$ -space  $N_x$ , for every arrow  $x : x \rightarrow y$ , there is given a linear transformation  $N_\alpha : N_x \rightarrow N_y$ . A representation  $N$  of  $Q$  over  $K$  (or better, the corresponding direct sum  $\oplus_x N_x$ ) is just an arbitrary (finite-dimensional)  $KQ$ -module.

Let  $d_x$  be the dimension of  $N_x$ , and  $d = \sum_x d_x$ ; we call  $d$  the dimension of  $N$ . A basis  $\mathcal{B}$  of  $N$  is by definition a subset of the disjoint union of the various  $K$ -spaces  $N_x$ , such that for any vertex  $x$  the set  $\mathcal{B}_x = \mathcal{B} \cap N_x$  is a basis of  $N_x$ . Let us assume that such a basis  $\mathcal{B}$  of  $N$  is given. For any arrow  $\alpha : x \rightarrow y$ , we may write  $N_\alpha$ , as a  $(d_y \times d_x)$ -matrix whose rows are indexed by  $\mathcal{B}_y$  and whose columns are indexed by  $\mathcal{B}_x$ . We denote by  $N_{\alpha, \mathcal{B}}(b, b')$  the corresponding matrix coefficients, where  $b \in B_x, b' \in B_y$ , these matrix coefficients  $N_{\alpha, \mathcal{B}}(b, b')$  are defined by  $N_\alpha(b) = \sum_{b' \in \mathcal{B}} N_{\alpha, \mathcal{B}}(b, b')b'$ . By definition, the coefficient quiver  $\Gamma(N, \mathcal{B})$  of  $N$  with respect to  $\mathcal{B}$  has the set  $\mathcal{B}$  as set of vertices, and there is an arrow  $(\alpha, b, b')$  provided  $N_{\alpha, \mathcal{B}}(b, b') \neq 0$ . The choice of basis dramatically affects the shape of the coefficient quiver.

The use of coefficient quivers to describe module structure is exhibited in detail in Ringel's appendix to [4]. Coefficient quivers provide a more precise tool than Alperin's module diagrams introduced in [1]. In this paper, we use Alperin diagrams in the simplest cases in order to describe module structure, but in more complicated situations where Alperin diagrams do not tell the whole story, we also use coefficient quivers.

#### 4. FACETS AND ALCOVES FOR $G = \mathrm{SL}_3$

Henceforth we take  $G = \mathrm{SL}_3(K)$ . (Recall that  $K$  is an algebraically closed field of characteristic  $p > 0$ .) Unless otherwise stated we shall assume that  $p \geq 5$ .

4.1. The Coxeter number of  $G = \mathrm{SL}_3(K)$  is  $h = 3$ . Note that  $p = 5$  is the smallest prime  $p$  such that  $p \geq 2h - 2$ . We take  $T$  to be the subgroup of diagonal matrices in  $G$  and  $B$  to be the subgroup of lower triangular matrices, and write  $\alpha_1, \alpha_2$  for the simple roots. We also write  $\varpi_1, \varpi_2$  for the fundamental weights, defined by  $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$ . Recall that  $\{\varpi_1, \varpi_2\}$  is a  $\mathbb{Z}$ -basis of  $X$  and that  $\lambda = \lambda_1 \varpi_1 + \lambda_2 \varpi_2$  lies in  $X^+$  if and only if  $\lambda_1, \lambda_2 \in \mathbb{N}$ . We call a weight  $\lambda$  *restricted* if  $\lambda \in X_1$ ; the set  $X_1$  is also called the restricted region. We let  $E$  denote the natural module for  $\mathrm{SL}_3$  of highest weight  $\varpi_1$ , and  $E^*$  denote its dual of highest weight  $\varpi_2$ .

4.2. We fix some notation regarding the alcoves of the affine Weyl group action on the weights of  $\mathrm{SL}_3(K)$ . The bottom alcove  $C_1$  is defined by

$$C_1 = \{\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} X : 0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}.$$

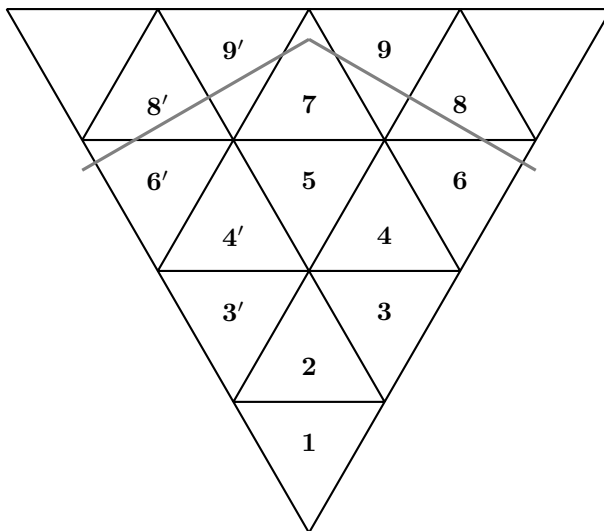
The affine Weyl group,  $W_p$ , is the group generated by the reflections in the walls of  $C_1$ . An arbitrary alcove is a translate of  $C_1$  under the action of  $W_p$ . In Figure 1 below we label the alcoves, which we call *fundamental* alcoves, that arise in the study of the family  $\mathfrak{F}(\mathrm{SL}_3)$ .

For each alcove label  $i$  we let integers  $(n_\alpha^i)$ , indexed by the set  $\{\alpha\}$  of positive roots, be determined so that  $C_i = \{\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} X : n_\alpha^i < \langle \lambda + \rho, \alpha^\vee \rangle < (n_\alpha^i + 1)p, \text{ for all } \alpha\}$  denote the alcove labelled  $i$ . The closure,  $\overline{C}_i$ , of  $C_i$  is then given by

$$\overline{C}_i = \{\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} X : n_\alpha^i \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq (n_\alpha^i + 1)p, \text{ for all } \alpha\}.$$

We also let  $\mathcal{F}_{i|j} = \overline{C}_i \cap \overline{C}_j$  denote the wall between two adjacent alcoves  $C_i$  and  $C_j$ ; this wall is a facet in the sense of [15, II.6.2].

*Notation.* We assume the reader is familiar with Jantzen's translation principle [15], which in particular implies an equivalence of module structure for highest weight modules belonging to the same facet. We will therefore adopt facet notation for highest weight modules throughout this paper, which replaces the highest weight  $\lambda \in X^+$  by its corresponding alcove label  $j$  whenever  $\lambda \in C_j$ . Thus, for example, for any  $\lambda \in C_1 \cap X$ ,  $L(\lambda)$  is denoted by


 FIGURE 1. The fundamental alcoves for  $G = \mathrm{SL}_3$ 

$L(1)$ ,  $\Delta(\lambda)$  is denoted by  $\Delta(1)$ ,  $T(\lambda)$  is denoted by  $T(1)$ , and so on. Furthermore, within module diagrams we will always identify simple modules  $L(j)$  with their alcove label, and thus will denote  $L(j)$  by simply  $j$ . For  $p$ -singular weights  $\lambda \in \mathcal{F}_{i,j}$  lying on the wall between two alcoves  $i$  and  $j$  we use the notation  $i|j$  to denote the facet, and use the notation  $L(i|j)$ ,  $\Delta(i|j)$ ,  $T(i|j)$  for the highest weight modules  $L(\lambda)$ ,  $\Delta(\lambda)$ ,  $T(\lambda)$  respectively. Furthermore, within module diagrams we will identify a simple module with its alcove label, thus denoting  $L(i|j)$  simply by  $i|j$ .

4.3. There is an involution on  $G$ -modules which on weights is the map  $\lambda \rightarrow -w_0(\lambda)$ , where  $w_0$  is the longest element of the Weyl group. (In Type  $A$  this comes from a graph automorphism of the Dynkin diagram.) We refer to this involution as *symmetry*, and omit results that can be obtained ‘by symmetry’. In the case  $G = \mathrm{SL}_3$ , this involution interchanges alcoves labelled  $j$  and  $j'$  in the numbering system introduced above.

## 5. THE STRUCTURE OF CERTAIN WEYL MODULES

In [17] the  $p$ -filtration structure of Weyl modules for  $\mathrm{SL}_3(K)$  is determined for all primes. When these layers are semisimple this gives the radical structure of the Weyl modules. Therefore when  $p \geq 5$  and we consider weights from the first  $p^2$ -alcove this re-derives the generic structures calculated in [12] and [14], which we shall recall below. The  $p$ -filtrations are semisimple for all but three of the Weyl modules considered in [4] and so provide an alternative proof of their structures. For a given prime these calculations can also be checked using the Weyl module GAP package available on the second author’s web page.

We remind the reader of the notational conventions of 4.2 and in particular that in diagrams we will identify simple modules with their facet label. The structure of the  $p$ -singular Weyl modules in question is given by the following strong Alperin diagrams, where as in [11], [4] we use the notation  $[L_1, L_2, \dots, L_s]$  to depict the structure of the unique uniserial module  $M$  with composition factors  $L_1, \dots, L_s$  arranged so that  $\mathrm{rad}_i M \simeq L_i$  for

all  $i$ .

$$\begin{aligned}
\Delta(1|2) &= [(1|2)], & \Delta(2|3) &= [(2|3)], \\
\Delta(3|4) &= [(3|4), (2'|3')], & \Delta(4|6) &= [(4|6), (1|2)], \\
\Delta(4|5) &= [(4|5), (3'|4'), (2|3)], & \Delta(6|8) &= [(6|8), (4|5)], \\
\Delta(8|9) &= [(8|9), (5|7), (4|6)], \\
\Delta(5|7) &= \begin{array}{c} (5|7) \\ \swarrow \quad \searrow \\ (4'|6') \quad (4|6) \\ \swarrow \quad \searrow \\ (1|2) \end{array}, & \Delta(7|9) &= \begin{array}{c} (7|9) \\ \swarrow \quad \searrow \\ (6|8) \quad (3'|4') \\ \swarrow \quad \searrow \\ (4|5) \end{array}.
\end{aligned}$$

The structure of the  $p$ -regular Weyl modules we need is as follows, where once again each diagram is a strong Alperin diagram.

$$\Delta(1) = [1], \quad \Delta(2) = [2, 1], \quad \Delta(3) = [3, 2], \quad \Delta(6) = [6, 4, 1],$$

$$\Delta(4) = \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 3 \quad 1 \quad 3' \\ \swarrow \quad \searrow \\ 2 \end{array}, \quad \Delta(8) = \begin{array}{c} 8 \\ \swarrow \quad \searrow \\ 3 \quad 5 \quad 6 \\ \swarrow \quad \searrow \\ 4 \end{array},$$

$$\begin{aligned}
\Delta(5) &= \begin{array}{c} 5 \\ \swarrow \quad \searrow \\ 4 \quad 1 \quad 4' \\ \swarrow \quad \searrow \\ 3 \quad 1 \quad 3' \\ \swarrow \quad \searrow \\ 2 \end{array}, & \Delta(7) &= \begin{array}{c} 7 \\ \swarrow \quad \searrow \\ 6 \quad 3 \quad 5 \quad 3' \quad 6' \\ \swarrow \quad \searrow \\ 4 \quad 1 \quad 4' \\ \swarrow \quad \searrow \\ 1 \end{array}, & \Delta(9) &= \begin{array}{c} 9 \\ \swarrow \quad \searrow \\ 8 \quad 7 \\ \swarrow \quad \searrow \\ 3 \quad 5 \quad 6 \quad 3' \\ \swarrow \quad \searrow \\ 4 \end{array}.
\end{aligned}$$

All these Weyl modules, including the  $p$ -singular ones, are rigid.

## 6. MAIN RESULTS

In this section we give the two main results of the paper. The first main result is a near classification of the members of the family  $\mathfrak{F} = \mathfrak{F}(\mathrm{SL}_3)$  of indecomposable direct summands of a tensor product of two restricted simple modules for  $\mathrm{SL}_3$ . The second main result is a description of the structure of all modules from the family  $\mathfrak{F}$  for  $p \geq 5$ . (The cases  $p = 2, 3$  were treated in [4].)

It turns out that the members of  $\mathfrak{F}(\mathrm{SL}_3)$  are not all highest weight modules. The highest weight modules in  $\mathfrak{F}(\mathrm{SL}_3)$  are either simple modules  $L(\lambda)$  or indecomposable tilting modules  $T(\lambda)$  for various  $\lambda$  described below. There is, up to isomorphism, a unique non highest weight module corresponding to each weight  $\lambda$  in the second alcove  $C_2$ , which we shall denote by  $M(\lambda)$ . For  $\lambda \in C_2$ , the module  $M(\lambda)$  has a simple socle and head isomorphic to  $L(\lambda)$ , with the quotient  $\mathrm{rad} M(\lambda) / \mathrm{soc} M(\lambda)$  of the radical by the socle a semisimple module isomorphic to  $L(w_1 \cdot \lambda) \oplus L(w_2 \cdot \lambda) \oplus L(w_3 \cdot \lambda)$ , where  $w_1, w_2, w_3$  are the elements of the affine Weyl group  $W_p$  corresponding to reflections in the three walls of the alcove  $C_2$ . Thus, we can depict the strong Alperin diagram of  $M(\lambda)$ , for any  $\lambda \in C_2$ , as follows

$$\begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 3 \quad 1 \quad 3' \\ \swarrow \quad \searrow \\ 2 \end{array}$$



using our convention of replacing weights by their alcove labels. Notice that  $M(\lambda)$  is rigid and self-dual under contravariant duality.

**Theorem 1.** *The family  $\mathfrak{F}(\mathrm{SL}_3)$ , up to isomorphism, consists of the following indecomposable modules:*

- (a)  $T(\lambda)$  for all  $\lambda \in X$  such that  $0 \leq \langle \lambda, \alpha^\vee \rangle \leq 2p - 2$  for all simple roots  $\alpha$ .
- (b)  $L(\lambda)$  and  $M(\lambda)$  for all  $\lambda \in C_2$ .
- (c) A finite list, depending on  $p$ , of ‘exceptional’ tilting modules of the form  $T(\lambda)$ , for various  $\lambda$  not already listed in part (a). For  $p = 2$  there are no exceptional tilting modules, for  $p = 3$  there are precisely four (see [4]) of highest weight lying on the boundary of  $\overline{C}_6$  and for larger  $p$  the number of exceptional modules grows with  $p$  with the highest weight of such modules lying in the region  $\overline{C}_6 \cup \overline{C}_8 \cup \overline{C}_9$  (and those obtained by symmetry).

Moreover, one can explicitly determine the decomposition of a tensor product of two  $p$ -restricted irreducible modules into members of  $\mathfrak{F}$ . The algorithm for this is outlined in Section 8.

Note that in case  $p = 2$  the alcove  $C_2$  is empty and so part (b) of the theorem is vacuous, so for  $p = 2$  the members of  $\mathfrak{F}$  are just the tilting modules listed in part (a). See [11] for a description of  $\mathfrak{F}(\mathrm{SL}_2)$ . The following diagrams illustrate the restrictions in the small primes cases.

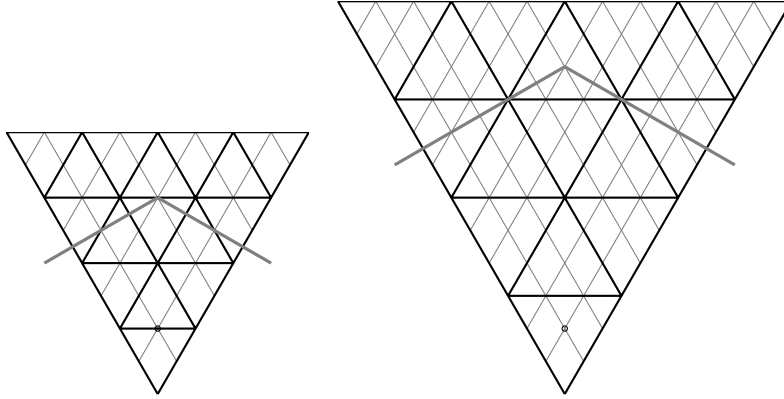


FIGURE 2. Alcoves in the region of Figure 1 for  $p = 2, 3$ . The intersection points of the light lines are integral weights, and the heavy lines define the walls of the alcoves. The circled point is the origin in the Euclidean plane, while the bottom vertex in each figure is the point  $-\rho$ .

For  $p \geq 3$  there are three vertices (points common to the closure of six alcoves) in the admissible region of weights defined in Figure 1, and each of them gives the highest weight of a (simple) tilting member of  $\mathfrak{F}(\mathrm{SL}_3)$ . These are the Steinberg module  $\mathrm{St} = T((p-1)\rho) = L((p-1)\rho)$  and the two modules  $T(p\varpi_1 + (p-1)\rho) = L(p\varpi_1 + (p-1)\rho) \simeq E \otimes \mathrm{St}$ ,  $T(p\varpi_2 + (p-1)\rho) = L(p\varpi_2 + (p-1)\rho) \simeq E^* \otimes \mathrm{St}$ . The next result gives information on the structure of the other tilting members of  $\mathfrak{F}(\mathrm{SL}_3)$ , of highest weight lying in alcoves or on walls between a pair of alcoves.

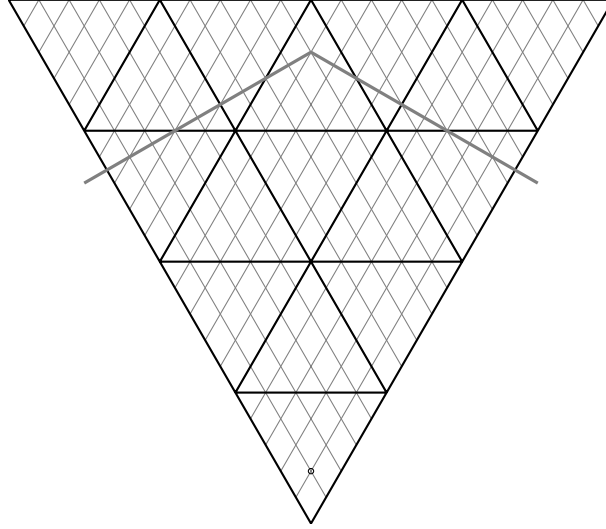


FIGURE 3. Alcoves in the region of Figure 1 for  $p = 5$ , with the same conventions as in Figure 2.

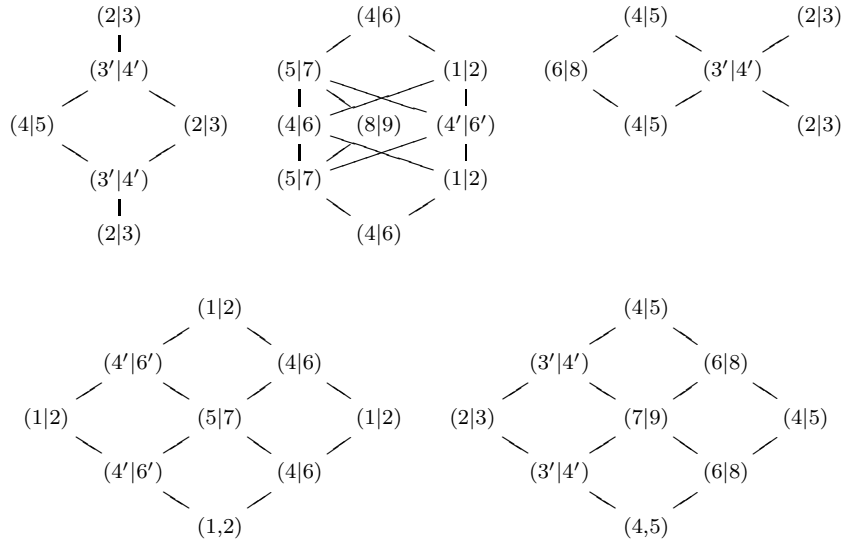
**Theorem 2.** *Let  $G = \mathrm{SL}_3(K)$  where the characteristic of  $K$  is  $p \geq 5$ .*

(a) *The  $p$ -singular tilting modules in  $\mathfrak{F}(\mathrm{SL}_3)$  of highest weight lying on walls are all rigid, with structure as follows. The uniserial modules of highest weight lying on walls are:*

$$\begin{aligned} T(1|2) &= [(1|2)]; & T(2|3) &= [(2|3)]; \\ T(3|4) &= [(2'|3'), (3|4), (2'|3')]; & T(4|6) &= [(1|2), (4|6), (1|2)] \end{aligned}$$

*along with their symmetric versions.*

*The non-uniserial modules of highest weight lying on walls are  $T(4|5)$ ,  $T(8|9)$ , and  $T(6|8)$ ,  $T(5|7)$  and  $T(7|9)$ , with structure given by the following strong Alperin diagrams, respectively:*



*along with their symmetric versions.*

$$T(1) = [1]; \quad T(2) = [1, 2, 1].$$

Note that all members of  $\mathfrak{F}(\mathrm{SL}_3)$  are local (and co-local) with simple  $p$ -restricted  $G_1T$ -socle except for  $T(8)$  and  $T(6|8)$  for  $p \geq 5$ . Therefore every direct summand in the decomposition (1.1.3) of [4] is indecomposable, unless it involves a factor of the form  $T(\lambda)$ , for  $\lambda \in C_8 \cup \mathcal{F}_{6|8}$ . Because of the upper bound constraint of  $2(p-1)\rho$  on the highest weights of tilting members of  $\mathfrak{F}(\mathrm{SL}_3)$ , we note also that when  $p = 5$  there are no members  $T(\lambda)$  in  $\mathfrak{F}(\mathrm{SL}_3)$  with  $\lambda \in C_8$  or  $C_9$  although such modules do appear for larger primes  $p > 5$ .

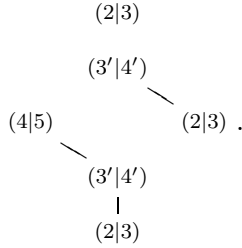
The proof of these theorems is given in the next two sections. In Section 7 we consider the structure of the relevant tilting modules. In Section 8 we consider which members of

$\mathfrak{F}$  appear in a given tensor product, and show that the modules  $M(\lambda)$  for  $\lambda \in C_2$  are the only non-simple and non-tilting modules in  $\mathfrak{F}$ .

## 7. THE TILTING MODULES

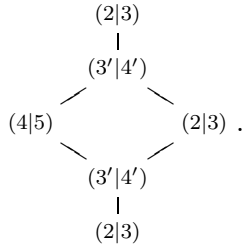
In computing the structure of tilting modules, our approach is along the finite-dimensional algebra lines of Ringel's appendix to [4]. This is a particularly useful approach for  $\mathrm{SL}_3(K)$  as most tilting modules we consider shall be projective for some generalised Schur algebra. We begin with the  $p$ -singular tilting modules. This is because their structures tend to be less complicated, and applying translation functors to them give useful filtrations of the  $p$ -regular tilting modules.

7.1. We shall build the tilting modules  $T(2|3), T(3'|4')$  and  $T(4|5)$  as modules for the Schur algebra  $S(\pi)$  corresponding to the poset  $\pi = \{(2|3) < (3'|4') < (4|5)\}$ . By 2.7 the tilting module  $T(4|5)$  is the projective cover of  $L(2|3)$ . By Brauer–Humphrey's reciprocity (2.5.1) and the Weyl module structure, the projective module  $P(2|3)$  has a  $\Delta$ -filtration with  $\Delta$ -factors  $\Delta(2|3)$ ,  $\Delta(3'|4')$ , and  $\Delta(4|5)$  each occurring with multiplicity one. Using Theorem 2.5 we can locate where the heads of the  $\Delta$ -modules occur in a radical filtration of the module. The diagram below gives the radical structure of the module, where we have highlighted the  $\Delta$ -filtrations.



We shall briefly highlight how this works: we have that  $\mathrm{rad}_1(\Delta(2|3)) = L(2|3)$ ,  $\mathrm{rad}_2(\Delta(3'|4')) = L(2|3)$  and  $\mathrm{rad}_3(\Delta(4|5)) = L(2|3)$ . Therefore the heads of the  $\Delta$ -modules  $\Delta(2|3)$ ,  $\Delta(3'|4')$ , and  $\Delta(4|5)$  appear in the first, second, and third layer of  $P(2|3)$  respectively (as pictured above). Note that  $\mathrm{rad}_4(P(2|3)) = L(3'|4')$ , however this module is not the head of a  $\Delta$ -module in a  $\Delta$ -filtration, even though the modules are isomorphic.

Considering also the  $\nabla$ -filtration gives us the full structure of the module to be:



The tilting module is projective-injective and so Theorem 2.5 (and the remark following) gives both the radical and socle structure; thus showing that the module is rigid and that the above is therefore a strong Alperin diagram. We note that this gives us the structures of  $T(2|3)$  and  $T(3'|4')$  as they both appear as quotient modules. To see this, notice that the module  $P(2|3)$  has a uniserial quotient module isomorphic to  $[(2|3), (3'|4'), (2|3)]$ ; this

quotient has both  $\Delta$  and  $\nabla$  filtrations, hence is tilting, and thus by highest weight considerations is isomorphic to  $T(3'|4')$ . Furthermore,  $L(2|3)$  must equal  $\Delta(2|3)$  by the strong linkage principle, so it is a (simple) tilting quotient as well.

We now determine the structure of  $T(6|8)$ . The calculation is similar to that given above. We have that  $\nabla(6|8) = [(6|8), (4|5)]$  and so we see that  $\Delta(4|5)$  must appear at the top of the module. Contravariant duality and our knowledge of the other Weyl modules in this block then give us the  $\Delta$ -filtration of the tilting module:

$$\begin{array}{ccccc} & (4|5) & & (2|3) & \\ & \searrow & & & \\ (6|8) & & (3'|4') & & \\ & \searrow & & \searrow & \\ & (4|5) & & (2|3) & \end{array} .$$

Consideration of the  $\nabla$ -filtration then gives us the full structure of the module.

The remaining tilting module from this block is  $T(7|9)$ . There are two ways to prove that this module is projective-injective for a generalised Schur algebra. Donkin's tilting tensor product theorem implies that, as an  $S(\leq (7|9))$ -module, this tilting module has simple head,  $L(4|5)$ , and therefore there exists a surjection  $P(4|5) \rightarrow T(7|9)$ . The same theorem allows us to calculate the character of  $T(7|9)$ , and the reciprocity rule (2.5.1) allows us to see that this character coincides with that of the projective module  $P(4|5)$  - therefore the map is injective. Alternatively, this follows from [6, Theorem 5.1]. Therefore  $T(7|9)$  can be seen to be rigid by Theorem 2.5. The full diagram can easily be deduced from the  $\Delta$  and  $\nabla$ -filtrations.

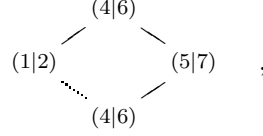
7.2. We now describe the structure of modules in the block indexed by  $(1|2)$  as modules for the Schur algebra  $S(\leq (5|7))$ . By 2.7 we have that  $T(5|7) \simeq P(1|2)$ . By Theorem 2.5 and the known structure of the Weyl modules it follows that  $T(5|7)$  has the following  $\Delta$ -filtration:

$$\begin{array}{ccccc} & (1|2) & & & \\ & \searrow & & (4'|6') & \\ (1|2) & & (4|6) & & (1|2) \\ & \searrow & & \searrow & \\ & (5|7) & & & \\ & \searrow & & \searrow & \\ (4|6) & & (4'|6') & & \\ & \searrow & & \searrow & \\ & (1|2) & & & \end{array}$$

The third Loewy layer is not multiplicity free, and so the complete structure of the diagram is not fully determined by consideration of the  $\nabla$ -filtration. We consider filtrations of  $T(4|5) \otimes E^*$  and  $T(4'|5) \otimes E$  and use translation functors in order to understand the structure of  $T(5|7)$ . Actually, to be precise, we pick the unique weight  $\lambda$  on the wall  $\mathcal{F}_{4|5}$  such that  $T(\lambda) \otimes E^*$  has highest weight on the wall  $\mathcal{F}_{5|7}$ , but we shall continue to use the facet notation below in the argument, in order to avoid switching back and forth between the two notations.

We first consider  $L(4|5) \otimes E^*$ . By Young's rule and our knowledge of Weyl characters we can compute the character of this tensor product to be  $\chi_p(5|7) + 2\chi_p(4|6) + \chi_p(1|2)$ . This module is a subquotient of the projective above, and so we know most of the diagram is

given as follows:



where the dotted line denotes the one extension yet to be determined. However, the tensor product of two simple modules is contravariantly self dual, and so the full diagram of the module (and its symmetric cousin) are as follows:

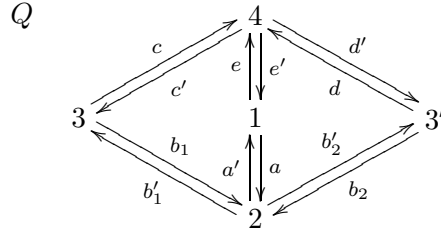
$$L(4|5) \otimes E^* = \begin{array}{ccc} & (4|6) & \\ (1|2) & \swarrow \quad \searrow & (5|7) \\ & (4|6) & \end{array} \quad L(4'|5) \otimes E = \begin{array}{ccc} & (4'|6') & \\ (1|2) & \swarrow \quad \searrow & (5|7) \\ & (4'|6') & \end{array}$$

We have that  $T(5|7)$  is a direct summand of  $T(4|5) \otimes E^*$  and of  $T(4'|5) \otimes E$ . Therefore  $L(4'|5) \otimes E$  and  $L(4|5) \otimes E^*$  appear as subquotients in  $T(5|7)$ . This allows us to deduce the remaining extensions in the Alperin diagram of  $T(5|7)$ , as given in Theorem 2.

As before, the structure of the smaller tilting modules in this block can be read off as quotient modules. The last tilting module in this block is  $T(8|9)$ , which is projective by similar arguments to those used for  $T(7|9)$  in subsection 7.1. Theorem 2.5 then gives the Loewy structure and implies that this module is rigid, similar to the previous cases. The Loewy layers are multiplicity-free and therefore consideration of  $\Delta$  and  $\nabla$ -filtrations gives the full diagram structure.

7.3. Having determined the structure of the  $p$ -singular tilting modules, we turn to the  $p$ -regular ones. The structure of the tilting modules  $T(1)$ ,  $T(2)$ , and  $T(3)$  is easily verified using the above techniques, but  $T(4)$  is more complicated and thus a detailed understanding of its structure can be reached only by studying the coefficient quivers (see Section 3) of projective modules for the generalised Schur algebra  $S(\leq 4)$ . The tilting module  $T(4)$  has a structural symmetry arising from the type  $A_2$  Dynkin diagram automorphism. It is the smallest tilting module of  $\mathrm{SL}_3$  exhibiting such a symmetry.

The dimension of  $\mathrm{Ext}_G^1$  between simple modules for an algebraic group  $G$  is determined by the structure of the Weyl modules (see Proposition II.4.14 in [15]). Therefore from the structure diagrams in Section 5 it follows that the  $\mathrm{Ext}^1$ -quiver  $Q$  for  $S(\leq 4)$  is as follows:



By Gabriel's theorem (see e.g. Proposition 4.1.7 in [3]) we have that  $S(\leq 4)$  is a quotient of the path algebra of the quiver  $Q$ .

Before considering the defining relations of  $S(\leq 4)$  we first consider the simpler question of describing the Schur algebra  $S' = S(1, 2, 3, 3')$ , which is a quotient algebra of  $S(\leq 4)$ , by quiver and relations. From the Weyl module structure, the Ext quiver for  $S'$  is the

full subquiver  $Q' = Q(1, 2, 3, 3')$  of  $Q$ , obtained by removing the vertex 4 and the arrows  $c, c', d, d', e, e'$ . We have the following description of the algebra  $S'$ .

**Proposition.** *The Schur algebra  $S' = S(1, 2, 3, 3')$  is isomorphic to the path algebra of  $Q'$  modulo the ideal generated by the following relations:*

$$ab'_1 = ab'_2 = 0, \quad aa'a = 0, \quad b_1a' = b_1b'_1 = b_1b'_2 = 0, \quad b_2a' = b_2b'_1 = b_2b'_2 = 0, \\ a'a = b'_1b_1 + b'_2b_2.$$

*Proof.* From the structure of the Weyl modules, the reciprocity (2.5.1), and Theorem 2.5 it follows that  $P(1)$ ,  $P(3)$ , and  $P(3')$  are uniserial with structure  $P(1) = [1, 2, 1]$ ,  $P(3) = \Delta(3) = [3, 2]$ ,  $P(3') = \Delta(3') = [3', 2]$ . This implies the relations in the first displayed line of the proposition.

We now address the remaining relation  $a'a = b'_1b_1 + b'_2b_2$ . It is immediate that  $a'a = \beta_1b'_1b_1 + \beta_2b'_2b_2$  where  $\beta_1, \beta_2$  are scalars, since otherwise the independence of the paths  $a'a$ ,  $b'_1b_1$ ,  $b'_2b_2$  would force  $[P(2): L(2)] > 3$ , which is a contradiction.

We have that  $\dim_K \text{Hom}_G(P(2), T(3)) = 2$  by Proposition 2.6, also  $\text{soc } T(3) = L(2)$ . Theorem 2.5 then gives the Loewy structure of  $P(2)$ . Consideration of the Loewy structure gives that one of the two homomorphisms is given by projection of the head of  $P(2)$  onto the socle of  $T(3)$ , and the other homomorphism is a surjection of  $P(2)$  onto  $T(3)$ . Therefore  $T(3)$  is a quotient module of  $P(2)$  and  $\beta_1 \neq 0$ . We have that  $\beta_2 \neq 0$  by symmetry.

Fixing our choice of  $a, a', b_1, b_2$  and adjusting our choice for  $b'_1, b'_2$  if necessary, we can pick  $\beta_1$  and  $\beta_2$  to both be 1.  $\square$

*Remark.* The assertion of the proposition can be visualised by drawing the shape of the indecomposable projective  $S'$ -modules. Each of the projective modules  $P(1), P(3)$ , and  $P(3')$  has a unique coefficient quiver as follows.

$$\begin{array}{ccc} 1 & 3 & 3' \\ | & | & | \\ 2 & 2 & 2 \\ | & & \\ 1 & & \end{array}.$$

The projective module  $P(2)$  has three different coefficient quivers:

$$\begin{array}{ccc} & 2 & \\ & | & \\ 3 & 1 & 3' \\ | & / & \backslash \\ 2 & & 2 \end{array} \quad \begin{array}{ccc} & 2 & \\ & | & \\ 3 & 1 & 3' \\ | & / & \backslash \\ 2 & & 2 \end{array} \quad \begin{array}{ccc} & 2 & \\ & | & \\ 3 & 1 & 3' \\ | & / & \backslash \\ 2 & & 2 \end{array}.$$

*Remark.* Using alcove instead of weight notation, we let  $M(2)$  denote the quotient module

$$M(2) = P(2)/(b'_1b_1 - b'_2b_2 = 0).$$

Then  $M(2)$  is the unique contravariantly self-dual non-tilting quotient of  $P(2)$ . For any  $\lambda \in C_2$  the module  $M(\lambda)$  defined earlier has the same structure as  $M(2)$ .

We now consider the Schur algebra  $S(\leq 4)$ . Applying appropriate translation functors (which are exact functors) to the embedding  $T(3|4) \hookrightarrow T(4'|5)$  produces an embedding  $T(4) \hookrightarrow T(5)$ ; see II.E.11 of [15]. By subsection 2.6 we have that  $T(5)$  is projective-injective and therefore  $T(4)$  has a simple head  $L(2)$ . The character of  $T(4)$  (computed using translation functors) is equal to that of the  $S(\leq 4)$  module  $P(2)$  (computed by BGG reciprocity) and therefore we have that  $P(2) = T(4)$  for  $S(\leq 4)$ .

We now use a combination of: the description from Theorem 2.5 of the Loewy structure of  $P(2)$ ; the coefficient quiver techniques for describing  $P(2)$ ; and filtrations of  $T(4)$  arising from tensor products of other tilting modules in order to describe the tilting module  $T(4)$ . Theorem 2.5 gives us the following filtration of our module:

$$(\dagger) \quad \begin{array}{ccccc} & & 2 & & \\ & & | & & \\ 3 & & 1 & & 3' \\ | & & & & | \\ 2 & & 4 & & 2 \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} .$$

We will need to consider two other filtrations of the module. One is obtained from  $E \otimes T(3|4)$ , the other is from  $L(2) \otimes T(3)$ . (As before, we must choose a representative  $T(\lambda) \in T(3|4)$  for an appropriate  $\lambda \in \mathcal{F}_{3|4}$ ; for  $p \geq 5$  there is more than one possible choice for  $\lambda$ .) We make use of the structure of the module  $M(2)$  defined in the remark above. We have, by consideration of the Loewy layers of  $T(3)$ , that  $L(2) \otimes T(3)$  can be filtered as follows

$$\begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} \oplus \begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} .$$

We know that  $\dim_K \operatorname{Hom}_G(L(2) \otimes T(3), L(4)) = \dim_K \operatorname{Hom}_G(T(3), L(4) \otimes L(2)) = 0$ , where the second equality follows from the Steinberg tensor product theorem. By self-duality and the  $(\dagger)$ -filtration we have that  $L(2) \otimes T(2) = T(4) \oplus L(2)$ . Thus we have that  $T(4)$  has filtration:

$$(\dagger\dagger) \quad \begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ | & \swarrow & & \searrow & | \\ 2 & & 4 & & 2 \\ & \swarrow & | & \searrow & \\ 3' & & 1 & & 3 \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} .$$

We can also filter  $T(4)$  by considering the tensor product  $E \otimes T(3|4)$  (after choosing an appropriate representative). We have, by consideration of the Loewy layers of  $T(3|4)$ , that  $E \otimes T(3|4)$  can be filtered as follows

$$(\dagger\dagger\dagger) \quad \begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ & \searrow & | & \swarrow & \\ 2 & & 4 & & 2 \\ | & \swarrow & & \searrow & | \\ 3' & & 1 & & 3 \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} .$$



We have that  $\dim_K(\text{Ext}_G^1(L(1), L(2))) = 1$ , therefore the  $(\dagger\dagger)$ -filtration implies the relation  $a'aa' = (b'_1b_1 + b'_2b_2)a' = 0$ . Therefore, if we let the coefficient of  $b_1b'_1a$  be 1 we have that the coefficient,  $\gamma$ , of  $b_2b'_2a$  is  $-1$ . In terms of the matrix presentation  $N$  of the projective module  $P(2) \simeq T(4)$  this can be expressed as

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = 0.$$

Comparing the two filtrations in light of  $\text{Ext}^1$  information we realise that  $b'_1b_1b'_2 = \mu_1a'ed'$ ,  $b'_1b_1b'_1 = 0$ ,  $b'_2b_2b'_1 = \mu_2a'ec'$  and  $b'_2b_2b'_2 = 0$ . We can then take  $\mu_1 = 1$ , which implies that  $\mu_2 = -1$  because

$$\begin{aligned} b'_1b_1b'_2b_2 - b'_2b_2b'_1b_1 &= (b'_1b_1 - b'_2b_2)(b'_1b_1 + b'_2b_2), \\ &= (b'_1b_1 - b'_2b_2)(a'a), \\ &= a'ee'a'. \end{aligned}$$

The coefficient quiver relative to this basis is as follows:

$$\begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ | & \times & | & \times & | \\ 2 & & 4 & & 2 \\ | & \times & | & \times & | \\ 3' & & 1 & & 3 \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array},$$

where there are two non-trivial coefficients  $N_{(b'_2b_2), (b'_2b_2a)} = -1 = N_{(b'_2b_2), (b'_2b_2b'_1)}$ .

**7.4. Alternative Bases for the Coefficient Quiver.** In [18] it is shown that any tilting module for a path algebra has a coefficient quiver such that all the coefficients can be chosen to be 1. There is no theory in place to tackle the problem for tilting modules of quotients of path algebras.

In this section we provide the two possible bases of coefficient quivers for the tilting module  $T(4)$ , such that all coefficients are 1. This is, to our knowledge, the first non-trivial example of such a coefficient quiver for a tilting module of an algebraic group. The interesting internal structure of the tilting module arises from the symmetry due to the graph automorphism of the type  $A_2$  Dynkin diagram. These bases and the basis from above are all possible coefficient quivers for the tilting module (modulo trivialities).

Consider the copy of  $L(2) \oplus L(2)$  in the third Loewy layer of our module. We note that in the coefficient quiver above we chose the basis to be  $\{b'_1b_1, b'_2b_2\}$ . The two other possible bases are  $\{a'a, b'_2b_2\}$  and  $\{a'a, b'_1b_1\}$  (where  $a'a = b'_1b_1 + b'_2b_2$ ). The coefficient quivers for these two bases, respectively, are:

$$\begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ | & \times & | & \times & | \\ 2 & & 4 & & 2 \\ | & \swarrow & | & \times & | \\ 3' & & 1 & & 3 \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array} \quad \begin{array}{ccccc} & & 2 & & \\ & \swarrow & | & \searrow & \\ 3 & & 1 & & 3' \\ | & \searrow & | & \times & | \\ 2 & & 4 & & 2 \\ | & \times & | & \searrow & | \\ 3' & & 1 & & 3 \\ & \searrow & | & \swarrow & \\ & & 2 & & \end{array}$$

We remark that the module  $T(4)$  has the same structure for  $p \geq 3$ , and that the diagram on the right is the description of this module given in [4]. The use of coefficient quivers has given us a more complete picture of the structure and symmetries of this module.

7.5. We now consider the module  $T(6)$ . This module can be seen to be projective-injective by application of translation functors to the embedding  $T(4|6) \hookrightarrow T(5|7)$ . Theorem 2.5 then gives us the Loewy structure of the  $S(\leq 6)$ -module  $P(1) \simeq T(6)$ . The interesting structure in the heart of the module comes from the fact that  $L(1)$  appears with multiplicity two, however this module is not as complicated as  $T(4)$ . The diagram is given by reconciling the sections  $\Delta(4)$ ,  $\Delta(2)$  with  $\nabla(4)$ ,  $\nabla(2)$  of the unique  $\Delta$  and  $\nabla$  filtrations. The coefficient quiver can be seen to be that described in Theorem 2 using the equalities  $\dim_K \text{Ext}_G^1(L(2), L(1)) = 1 = \dim_K \text{Ext}_G^1(L(4), L(1))$ .

7.6. We will need the following character formula, which may be proved by a straightforward application of the Littlewood–Richardson rule.

**Lemma.** *Let  $\Delta(a\varpi_1 + b\varpi_2)$  be a Weyl module of highest weight  $a\varpi_1 + b\varpi_2$ . Then  $\chi(E \otimes \Delta(a\varpi_1 + b\varpi_2)) = \chi((a+1)\varpi_1 + b\varpi_2) + \chi(a\varpi_1 + (b-1)\varpi_2) + \chi((a-1)\varpi_1 + (b+1)\varpi_2)$  provided that  $a, b \geq 1$ , and the same holds when  $a$  or  $b = 0$  if we simply ignore the characters corresponding to any non-dominant weights.*

7.7. In order to establish the character of the non-local module  $T(8)$  we consider  $E \otimes T(6|8)$ . (Once again, a choice of representative  $T(\lambda) \in T(6|8)$  must be made.) By applying Lemma 7.6 to the  $p$ -regular block of  $E \otimes T(6|8)$  we get the resulting character to be  $\chi(8) + \chi(6) + \chi(5) + \chi(4) + \chi(3) + \chi(2)$ . Straightforward Hom-calculations then give the socle and head of  $E \otimes T(6|8)$  to be  $L(4) \oplus L(2)$ . Knowledge of the other tilting characters implies that the only possible splitting of the tensor product is given by  $E \otimes T(6|8) = T(3) \oplus T(8)$ . However, if this is the case then  $T(8)$  is a quotient of  $P(4)$ ; we could then use Theorem 2.5 to describe its Loewy structure. The resulting module is easily seen to fail to be contravariantly self dual. Therefore  $E \otimes T(6|8) = T(8)$ .

We know that  $\Delta(3)$  must extend from  $\Delta(2)$  by Proposition 2.6 and therefore it is correctly placed within the Loewy structure. We then consider  $\text{Hom}_{S(\leq 8)}(P(4), T(8))$ . We have that  $P(4) \in \mathcal{F}(\Delta)$  and  $T(8) \in \mathcal{F}(\nabla)$ ; therefore by proposition 2.6 we have that  $\dim_K \text{Hom}_{S(\leq 8)}(P(4), T(8)) = 4$ . This allows us to place the Weyl modules within the Loewy structure of  $T(8)$  using Theorem 2.5.

7.8. We have that  $T(5)$ ,  $T(7)$ , and  $T(9)$  are projective for suitable Schur algebras by Proposition 2.6 (and in the case of  $T(9)$ , arguments similar to those in subsection 7.1). Therefore Theorem 2.5 gives their Loewy structures and proves that they are rigid, as claimed.

## 8. RESTRICTED TENSOR PRODUCT DECOMPOSITIONS

We now discuss the decompositions of restricted tensor products. We explain how to compute the character of a tensor product of simple  $p$ -restricted simple modules using the Littlewood–Richardson rule. We show that the summands in a decomposition of such a tensor product are all tilting modules, simple modules from the second alcove, and the modules  $M(\lambda)$ . We also show that the character of such a tensor product completely determines its decomposition into indecomposable summands, hence giving an algorithm

for the computation of the decomposition of the tensor product. In this section we freely switch between alcove and weight notation depending on our needs.

8.1. All restricted simple modules from  $C_1$  are tilting modules, and all restricted  $p$ -singular simple modules are also tilting modules. Therefore any tensor product of  $p$ -restricted simple modules not involving  $L(2)$  is a direct sum of tilting modules. We have that the character of the tensor product of two Weyl modules is determined by the Littlewood–Richardson rule. We know the characters of the tilting modules by Section 7, and so we can determine which modules appear as direct summands. This works by highest weight theory: take the highest weight appearing in the character, then subtract the tilting character corresponding to this highest weight, and repeat.

8.2. We have that modules of the form  $L(2)$  are the only non-tilting  $p$ -restricted simple modules. We first consider the tensor product of  $L(2)$  with a  $p$ -restricted simple tilting module,  $L$ . We have that any such  $L = L(r\varpi_1 + s\varpi_b)$  appears as a direct summand of  $E^{\otimes r} \otimes (E^*)^{\otimes s}$ . Therefore we consider  $L(2) \otimes E$  (and by symmetry  $E^*$ ). By Lemma 7.6 we have that  $L(2) \otimes E$  is a direct sum of simple modules from the upper closure of  $C_2$ . Translation functors are faithful within an alcove (therefore taking simple modules from  $C_2$  to simple modules from  $C_2$ ), so we only need check what happens on the walls.

The simple modules on the upper walls of  $C_2$  are tilting and so we inductively have that  $L(2) \otimes E^{\otimes r}$  is a direct sum of simple modules from  $C_2$  and tilting modules. We now require some further analysis of the splitting of tilting modules under translation functors.

**Lemma.** *Let  $\lambda$  be a  $p$ -singular weight from the first  $p^2$ -alcove. Then the projection onto the  $p$ -regular summand of  $T(\lambda) \otimes E$  is the indecomposable tilting module  $T(\lambda + \varpi_1)$  or zero.*

*Proof.* This is an observation using Lemma 7.6 and the characters of the tilting modules from the first  $p^2$ -alcove.  $\square$

We therefore have that  $L(2) \otimes E^{\otimes r} \otimes (E^*)^{\otimes s}$  is a direct sum of simple modules from  $C_2$  and tilting modules of highest weight  $> 2$ . Therefore the module  $L(2)$  is the unique module of highest weight 2 that may appear as a summand. Therefore the decomposition of the tensor product  $L(2) \otimes L$  is uniquely determined by its character.

8.3. We now consider the tensor product  $L(2) \otimes L(2)$ . We shall first show that the modules involved in the decomposition are tilting modules and the module  $M(\lambda)$  for  $\lambda \in C_2$ . We then give an explicit decomposition of a ‘minimal’ tensor product; translation functors may then be used to decompose an arbitrary tensor product of the form  $L(2) \otimes L(2)$ .

8.3.1. We note that  $L(2)$  is a submodule of both  $T(3)$  and  $T(3')$ . Therefore  $L(2) \otimes L(2)$  is a contravariantly self dual submodule of the tilting module  $T(3) \otimes T(3)$  and of  $T(3) \otimes T(3')$ .

We now restrict to consideration of a tensor product of irreducible modules of minimal highest weight from the alcove  $C_2$ . These correspond to the tilting modules of weight  $(p+a)\varpi_1 \in C_3$  and  $(p+b)\varpi_2 \in C_{3'}$  for  $a, b < p$ . We wish to consider the modules which appear as submodules in both tensor products  $(T(3) \otimes T(3))$  and  $T(3) \otimes T(3')$ ; by character arguments, these modules are tilting modules of highest weight  $\leq 4$ . We have that  $T(4)$  is the only such tilting module with a non-trivial contravariantly self-dual submodule, the module  $M$  described in the remark following Proposition 7.3. Therefore  $M$  is the unique

possible non-tilting module in this ‘minimal’ tensor product. Similar character theoretic arguments imply existence.

One then shows that translation functors take  $M$  to a sum of tilting modules with  $p$ -singular highest weights, and therefore that  $M$  is the only possible non-tilting summand of  $L(2) \otimes L(2)$ .

8.3.2. We now consider the tensor product of two simple modules with corresponding highest weights  $(p-2-a)\varpi_1 + (a+1)\varpi_2 \in C_2$  and  $(p-2-b)\varpi_1 + (b+1)\varpi_2 \in C_2$ , where  $a+b \leq p-1$  and  $b \leq a$ . We may restrict to this case due to symmetry. The simple tensor product appears as a quotient of the tensor product  $\nabla((p+a)\varpi_1) \otimes \nabla((p+b)\varpi_1)$ . The character of this tensor product is given by the Littlewood–Richardson rule as follows

$$\chi((p+b)\varpi_1) \cdot \chi((p+a)\varpi_1) = \sum_{i \leq b} \chi((2p+a+b-2i)\varpi_1 + i\varpi_2).$$

The projection onto any  $p$ -singular block is a direct sum of tilting modules and therefore determined by the character. We therefore project onto a  $p$ -regular block and consider the resulting module. We then calculate the following characters

$\chi((p+b)\varpi_1) \cdot \chi((p+a)\varpi_1)$	$\chi_p((p-2-b)\varpi_1 + (b+1)\varpi_2) \cdot \chi_p((p-2-a)\varpi_1 + (a+1)\varpi_2)$
$\chi(6) + \chi(4) + \chi(3')$	$\chi_p(3) + \chi_p(3') + 2\chi_p(1) + 2\chi_p(2)$
$\chi(6) + \chi(4)$	$\chi_p(2) + 2\chi_p(1)$
$\chi(3) + \chi(2)$	$\chi(3) + \chi(2)$

The tensor product of Weyl modules has a Weyl filtration and the tensor product of simple modules is contravariantly self dual. For each of the three cases above there exists a unique exact sequence of modules satisfying these conditions, given as follows:

$$\begin{array}{c}
 \begin{array}{c} 1 \\ | \\ 4 \\ | \\ 6 \end{array} \oplus \begin{array}{c} \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 3' \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 4 \quad 2 \\ \swarrow \quad \searrow \\ 3' \end{array} \end{array} \rightarrow \begin{array}{c} 1 \end{array} \oplus \begin{array}{c} \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 3' \quad 1 \quad 3 \\ \swarrow \quad \downarrow \quad \searrow \\ 2 \end{array} \end{array}
 \end{array}$$
  

$$\begin{array}{c} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 2 \quad 4 \\ \swarrow \quad \downarrow \quad \searrow \\ 1 \quad 3 \quad 6 \quad 3' \\ \swarrow \quad \downarrow \quad \searrow \\ 4 \end{array} \end{array} \rightarrow \begin{array}{c} 1 \\ | \\ 2 \\ | \\ 1 \end{array}$$

and in the third case the quotient is trivial, therefore the module has a Weyl filtration and is self dual and therefore tilting. We therefore have a complete decomposition of a tensor product of ‘minimal’ weight irreducible modules from the second alcove.

We let  $M'$  denote the module  $M \oplus L(1)$  appearing in the surjection above. We have seen that a simple module  $L(1)$  can only appear as a summand in a minimal tensor product if it appears as a summand of some  $M'$ .

Therefore, by Lemma 8.2, for an arbitrary tensor product  $L(2) \otimes L(2)$  we have that a simple module  $L(1)$  can only ever appear as a summand of a translate of some module  $M'$  (which is again an  $M'$  by translation).

Recall the characters of the modules  $M'$ ,  $T(3)$ ,  $T(3')$  and  $T(2)$  are as follows

$$\begin{aligned}\text{ch}(M') &= \chi_p(3) + \chi_p(3') + 2\chi_p(2) + 2\chi_p(1) \\ \text{ch}(T(3)) &= \chi_p(3) + 2\chi_p(2) + \chi_p(1) \\ \text{ch}(T(3')) &= \chi_p(3') + 2\chi_p(2) + \chi_p(1) \\ \text{ch}(T(2)) &= \chi_p(2) + 2\chi_p(1).\end{aligned}$$

These characters are linearly independent and therefore the decomposition of a tensor product  $L(2) \otimes L(2)$  is uniquely determined by its character.

**8.4. The algorithm.** Let  $L = L(\nu)$  and  $L' = L(\nu')$  be two  $p$ -restricted simple modules for  $G$ . The following algorithm will give a decomposition of the tensor product  $L \otimes L'$ .

- Use the Littlewood–Richardson rule to write out the character of  $L \otimes L'$  in terms of the basis given by the simple characters (as outlined case-by-case above).
- Rewrite the character in terms of the following bases in each case:
  - (1) If  $\nu, \nu' \notin C_2$  then the basis given by the tilting characters  $\text{ch}(T(\lambda))$ .
  - (2) If  $\nu \notin C_2, \nu' \in C_2$  then the basis given by  $\text{ch}(T(\lambda))$  for  $\lambda > 2$  and  $L(\mu)$  for  $\mu \in C_2$ .
  - (3) If  $\nu, \nu' \in C_2$  then the basis given by  $\text{ch}(T(\lambda))$  for  $\lambda \geq 2$  and  $M(\mu)$  for  $\mu \in C_2$ .

*Example* (Case 2). Let  $k$  be an algebraically closed field of characteristic 5. Take the tensor product  $L(2\varpi_1 + 2\varpi_2) \otimes L(\varpi_1 + \varpi_2)$ . The Littlewood–Richardson rule gives the character to be:

$$\chi_p(3(\varpi_1 + \varpi_2)) + \chi_p(\varpi_1 + 4\varpi_2) + \chi_p(4\varpi_1 + \varpi_2) + \chi_p(2(\varpi_1 + \varpi_2))$$

and this can be seen to give a direct sum decomposition by the linkage principle.

*Example* (Case 3). Let  $k$  be an algebraically closed field of characteristic 5. Take the tensor product  $L(3\varpi_1 + \varpi_2) \otimes L(3\varpi_1 + \varpi_2)$ . The Littlewood–Richardson rule gives the character to be:

$$\begin{aligned}&\chi_p(6\varpi_1 + 2\varpi_2) + 2\chi_p(2\varpi_1 + 4\varpi_2) + \chi_p(4\varpi_1 + 3\varpi_2) \\ &+ \chi_p(7\varpi_1) + \chi_p(5\varpi_2) + 2\chi_p(\varpi_1 + 3\varpi_2) + 2\chi_p(2\varpi_2)\end{aligned}$$

The  $p$ -singular characters  $\chi_p(4\varpi_1 + 3\varpi_2)$  and  $\chi_p(6\varpi_1 + 2\varpi_2) + 2\chi_p(2\varpi_1 + 4\varpi_2)$  are both tilting. This leaves us with one block with character  $\chi_p(7\varpi_1) + \chi_p(5\varpi_2) + 2\chi_p(\varpi_1 + 3\varpi_2) + 2\chi_p(2\varpi_2)$ . This is the character of  $M'$ . Therefore  $L(3\varpi_1 + \varpi_2) \otimes L(3\varpi_1 + \varpi_2) = M(\varpi_1 + 3\varpi_2) \oplus T(2\varpi_2) \oplus T(6\varpi_1 + 2\varpi_2) \oplus T(4\varpi_1 + 3\varpi_2)$ .

**8.5. Discussion.** For  $\text{SL}_2$  (see [11]) and again for  $\text{SL}_3$  in small characteristic (see [4]) the study of the family  $\mathfrak{F}$  gave a complete description of arbitrary simple tensor products. The reason that these cases are tractable is that all tilting modules in the range of  $\text{St} \otimes \text{St}$  have simple  $p$ -restricted socles or are factorisable by Donkin’s tilting tensor product theorem, therefore Lemma 1.1 of [4] is decisive in determining the decompositions in (1.1.3) of [4].

For  $\text{SL}_3$  in higher characteristics, we advance further into the alcove geometry. The tilting modules  $T(8)$  and  $T(6|8)$  which now appear have non-simple, non- $p$ -restricted socles, and cannot be factorised as they are too close to the walls. Such modules pose a substantial problem, and appear more frequently as we consider more complicated examples. For example in type  $B_2$  for all characteristics we get a tilting module with non-simple non- $p$ -restricted socle, as discussed in §5.8 of [2]. Therefore, for a general semisimple  $G$  the study

of the finite family  $\mathfrak{F}(G)$  of indecomposable direct summands of  $p$ -restricted tensor products cannot be expected to provide a complete solution to the problem of decomposing tensor products of two simples of arbitrary highest weights.

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